Vectors – Equation of lines

No calculators allowed.

1. The Cartesian equations of a line are \( \frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{3} \). Find the vector equation of the line.

2. The two lines, whose vector equations are given below, intersect. Find the point of intersection.

\[
r = \begin{pmatrix} 3 \\ 4 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \quad \quad r = \begin{pmatrix} -1 \\ 7 \\ 5 \\ 3 \end{pmatrix} + t \begin{pmatrix} 12 \\ 6 \end{pmatrix}
\]

3. The position vectors \( \vec{OA} \) and \( \vec{OB} \) are \( 2\mathbf{i} - \mathbf{j} + \mathbf{k} \) and \( \mathbf{i} + \mathbf{j} - \mathbf{k} \) respectively.

(a) Show that a vector equation for line \( (AB) \) can be written as \( \mathbf{i}(2 + \lambda) + \mathbf{j}(-1 - 2\lambda) + \mathbf{k}(1 + 2\lambda) \).

(b) There exists a point \( P \) on line \( (AB) \) such that \( \vec{OP} \) is perpendicular to \( (AB) \). Find the coordinates of \( P \).

(c) Hence, or otherwise, find the perpendicular distance from the origin to the line \( (AB) \).

4. Consider the two lines \( L_1 \) and \( L_2 \) with the following parametric equations:

\( L_1: \ x = -1 - 2\mu, \ y = \mu, \ z = 2 + 3\mu \) \quad \quad \( L_2: \ x = 2 + \lambda, \ y = -\lambda, \ z = 2 - \lambda \)

(a) Show that lines \( L_1 \) and \( L_2 \) are skew.

(b) A third line, \( L_3 \), has the direction vector \( \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \). Verify that \( L_3 \) is perpendicular to \( L_1 \) and \( L_2 \).

(c) Find parametric equations for \( L_3 \) given that it passes through the point \( A(1, 1, 3) \).

(d) Find the coordinates of the point \( B \) where \( L_1 \) and \( L_3 \) intersect.

5. Find the coordinates of the point on the line \( L \) (equation below) which is nearest to the origin.

\( L: \ x = 1 - \lambda, \ y = 2 + 3\lambda, \ z = 3 + \lambda \)

6. The points \( A, B \) and \( C \) have position vectors \( \mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \), \( \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} \) and \( \mathbf{c} = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} \) respectively.

(a) Find the position vector of the point \( P \) on line \( (BC) \) such that \( \vec{AP} \) is perpendicular to \( \vec{BC} \).

(b) Hence, or otherwise, find the shortest distance from \( A \) to the line \( (BC) \).
Vector – Scalar product and application

1. Find the exact measure of the angle between the vectors \( \mathbf{i} + \mathbf{k} \) and \( \mathbf{j} + \mathbf{k} \).
   \[ \text{[no calculator]} \]

2. Find the value(s) of \( c \) for which the vectors \( \begin{pmatrix} 10 \\ c \end{pmatrix} \) and \( \begin{pmatrix} c \\ 3c \end{pmatrix} \) are:
   (a) parallel;
   (b) perpendicular. \[ \text{[no calculator]} \]

3. Consider vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) such that \( \mathbf{a} \cdot \mathbf{c} = 3 \) and \( \mathbf{b} \cdot \mathbf{c} = 4 \). Given that the vector \( \mathbf{d} = \mathbf{a} + t \mathbf{b} \) is perpendicular to \( \mathbf{c} \), find the value of \( t \). \[ \text{[no calculator]} \]

4. The vectors \( \mathbf{a}, \mathbf{b} \) each have length 2. Given that \( (2\mathbf{a} - 3\mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} \)
   find the angle between \( \mathbf{a} \) and \( \mathbf{b} \). \[ \text{[calculator allowed]} \]
Angles, circles, arcs and sectors

1. A circle of radius 8 cm has a sector whose central angle has radian measure of $\frac{3}{7}$. Find the following exactly:
   (a) the length of the arc from A to B passing through C.
   (b) the area of the shaded sector.
   [no calculator]

2. O is the centre of a circle with radius 24 cm.
   Chord [GH] is 36 cm. Find the area of the shaded region.
   [calculator allowed]

3. The semi-circle with centre O shown at right has an area of exactly 24 cm$^2$.
   (a) Show that the shaded area can be expressed as $\frac{24\theta}{\pi} - \frac{24}{\pi}\sin\theta$.
   (b) If $\theta = \frac{2\pi}{3}$, find the exact area of the shaded region.
   [no calculator]

4. Two circles with the same radius $r$ intersect as shown.
   The angle subtended by the common chord (dashed in diagram) at the centre of each circle is $2\theta$.
   (a) Find an expression in terms of $r$ and $\theta$ for the shaded area.
   (b) If the shaded area is equal to $\frac{1}{4}$ of the area of one of the two circles show that $8\theta - 4\sin 2\theta = \pi$. Hence, find $\theta$ accurate to three significant figures.
   [calculator allowed]
Trigonometry

Part 1 – NO calculator allowed – Questions 1-7

1. Find all exact solutions to the equation $\sin 2\theta = \frac{1}{2}$ in the interval $0 \leq \theta \leq 2\pi$. [7 marks]

2. The diagram shows a circle with centre O. Angle AOB has a measure of $\frac{2\pi}{3}$ radians. The shaded region (a sector of the circle) has an area of $12\pi$ cm$^2$. Find the exact area of triangle AOB. [6 marks]

3. The diagram shows part of the graph of the function $y = a \sin bx$. Write down the values of $a$ and $b$. [4 marks]

4. Find all exact solutions to the equation $\cos 2x - \sin 2x = 1$ in the interval $0 \leq \theta < 360^\circ$. [6 marks]

5. If $\sin \alpha = -\frac{4}{5}$ and $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, find the exact values of the following: [6 marks]
   (a) $\cos \alpha$
   (b) $\tan \alpha$
   (c) $\sec 2\alpha$

6. (a) Find the exact value of $x$ in the diagram at right. [6 marks]

(b) Find the exact area of the triangle.

7. In triangle ABC, AB = 9 cm, AC = 12 cm, and angle B is twice the size of angle C. Find the exact value of the cosine of angle C; that is, value of $\cos C$, not the measure of angle C. [5 marks]

--- end of part 1 ---
Part 2 – Calculator allowed – Questions 8–12

8. Find all of the values of $\theta$ in the interval $0 \leq \theta \leq \pi$ that satisfy the equation $\tan 2\theta = \frac{4}{3}$. [4 marks]

9. Find the length(s) of AC in triangle ABC given that angle A is 42 degrees, AB=12.7 cm and BC=10.2 cm. [8 marks]

10. The depth of water, $h$ meters, measured at a sea pier $t$ hours after midnight is given by the function

$$h = a + b \cos \left( \frac{2\pi}{k} t \right),$$

where $a$, $b$ and $k$ are constants.

The water is at a maximum depth of 21 m at midnight and noon, and is at a minimum depth of 13 m at 06:00 and at 18:00.

Determine the values of

(a) $a$ 
(b) $b$ 
(c) $k$ [6 marks]

11. Prove the following identity:

$$\cot \theta + \tan \theta = \sec \theta$$

[5 marks]

12. In the diagram below, AD is perpendicular to BC. CD = 4, BD = 2 and AD = 3. $\angle CAD = \alpha$ and $\angle BAD = \beta$.

Show that the exact value of $\cos(\alpha - \beta) = \frac{17\sqrt{13}}{65}$. [4 marks]

Bonus Question

Find the exact value of $\tan \alpha$. [+3 marks]
Introduction to Differential calculus

Limits

1 Evaluate:
   a \( \lim_{x \to 3} (x + 4) \)  
   b \( \lim_{x \to -1} (5 - 2x) \)  
   c \( \lim_{x \to 4} (3x - 1) \)  
   d \( \lim_{x \to 2} (5x^2 - 3x + 2) \)  
   e \( \lim_{h \to 0} h^2(1 - h) \)  
   f \( \lim_{x \to 0} (x^2 + 5) \)

2 Evaluate:
   a \( \lim_{x \to 9} 5 \)  
   b \( \lim_{h \to 2} 7 \)  
   c \( \lim_{x \to 0} c, \ c \) a constant

3 Evaluate:
   a \( \lim_{x \to 1} \frac{x^2 - 3x}{x} \)  
   b \( \lim_{h \to 2} \frac{h^2 + 5h}{h} \)  
   c \( \lim_{x \to 0} \frac{x - 1}{x + 1} \)  
   d \( \lim_{x \to 0} \frac{x}{x} \)

4 At what values of \( x \) are the following functions not continuous? Explain your answer in each case.
   a \( f(x) = \frac{1}{x} \)  
   b \( f(x) = \frac{x^2 - x}{x} \)

5 Evaluate the following limits:
   a \( \lim_{x \to 0} \frac{x^2 - 3x}{x} \)  
   b \( \lim_{x \to 0} \frac{x^2 + 5x}{x} \)  
   c \( \lim_{x \to 0} \frac{2x^2 - x}{x} \)  
   d \( \lim_{h \to 0} \frac{2h^2 + 6h}{h} \)  
   e \( \lim_{h \to 0} \frac{3h^2 - 4h}{h} \)  
   f \( \lim_{h \to 0} \frac{h^3 - 8h}{h} \)  
   g \( \lim_{x \to 1} \frac{x^2 - x}{x - 1} \)  
   h \( \lim_{x \to 2} \frac{x^2 - 2x}{x - 2} \)  
   i \( \lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} \)

Limits at infinity

1 Examine \( \lim_{x \to \infty} \frac{1}{x^2} \).

2 Evaluate:
   a \( \lim_{x \to \infty} \frac{3x - 2}{x + 1} \)  
   b \( \lim_{x \to \infty} \frac{1 - 2x}{3x + 2} \)  
   c \( \lim_{x \to \infty} \frac{x}{1 - x} \)  
   d \( \lim_{x \to \infty} \frac{x^2 + 3}{x^2 - 1} \)  
   e \( \lim_{x \to \infty} \frac{x^2 - 2x + 4}{x^2 + x - 1} \)
The derivative function

1. a Find, from first principles, the gradient function of \( f(x) \) where \( f(x) \) is:
   i \( x \) ii \( 5 \)
   iii \( x^3 \) iv \( x^4 \)

   b Hence predict a formula for \( f'(x) \)
   where \( f(x) = x^n, \ n \in \mathbb{N} \).

2. Find \( f'(x) \) from first principles, given that \( f(x) \) is:
   a \( 2x + 5 \) b \( x^2 - 3x \) c \( -x^2 + 5x - 3 \)

3. Find \( \frac{dy}{dx} \) from first principles given:
   a \( y = 4 - x \) b \( y = 2x^2 + x - 1 \) c \( y = x^3 - 2x^2 + 3 \)

4. Use the first principles formula 
\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]
   to find:
   a \( f'(2) \) for \( f(x) = x^3 \) b \( f'(3) \) for \( f(x) = x^4 \).

5. Use the first principles formula to find the gradient of the tangent to:
   a \( f(x) = 3x + 5 \) at \( x = -2 \)
   b \( f(x) = 5 - 2x^2 \) at \( x = 3 \)
   c \( f(x) = x^2 + 3x - 4 \) at \( x = 3 \)
   d \( f(x) = 5 - 2x - 3x^2 \) at \( x = -2 \)

6. a Given \( y = x^3 - 3x \), find \( \frac{dy}{dx} \) from first principles.
   b Hence find the points on the graph at which the tangent has zero gradient.

7. Find \( \frac{dy}{dx} \) from first principles given:
   a \( y = \frac{4}{x} \) b \( y = \frac{4x + 1}{x - 2} \)

8. Use the first principles formula to find the gradient of the tangent to:
   a \( f(x) = \frac{1}{x^2} \) at \( x = 3 \)
   b \( f(x) = \frac{3x}{x^2 + 1} \) at \( x = -4 \)
   c \( f(x) = \sqrt{x} \) at \( x = 4 \)
   d \( f(x) = \frac{1}{\sqrt{x}} \) at \( x = 1 \)